# Penetration Depth and the Conductivity Sum Rule for a Model With Incoherent c-axis Coupling

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The conductivity sum rule for a one-band hopping model relates the integrated spectral weight of the real part of the conductivity to the average kinetic energy. For such a model, the superconducting penetration depth is therefore dependent upon both the change in the conductivity spectral weight and the change in kinetic energy between the normal and superconducting states. Here we examine the consequences of this for the c-axis penetration depth of a layered system in which the charge transfer perpendicular to the layers (along the c-axis) is mediated by interlayer impurity scattering.

The nature of the frequency and temperature dependent c-axis conductivity, $\sigma_{1c}(\omega, T)$, in the cuprate superconductors remains controversial, but for a number of these materials it appears to be weak and incoherent[1]. Recently, a simple model[2-4] consisting of layers with BCS quasiparticles which have a $d_{x^2-y^2}$ gap and an interlayer coupling mediated by impurity scattering was used to calculate $\sigma_{1c}(\omega, T)$. For this model, the conductivity sum rule relates the integrated spectral weight under $\sigma_{1c}(\omega, T)$ to the average kinetic energy per unit cell in the c-direction[5]. For such a model, the superconducting penetration depth is dependent upon both the change in the conductivity spectral weight and the change in the kinetic energy. Here we examine this and discuss its consequences.

We consider a Hamiltonian of the form

$$H = H_{ab} + H_c \tag{1}$$

where $H_{ab}$ describes the intralayer dynamics and $H_c$ is the interlayer coupling

$$H_c = \sum_{l,s} V_l(c^\dagger_{l+z,s}c_{l,s} + c^\dagger_{l,s}c_{l+z,s}) \tag{2}$$

Here $V_l$ is a random potential due to impurity scattering between layers. We assume that $H_{ab}$ describes quasiparticles with energy $\varepsilon_p$ in the normal state and BCS quasiparticles with dispersion $E_p = \sqrt{\varepsilon_p^2 + \Delta_p^2}$ in the superconducting state with $\Delta_p = \Delta_0 \cos 2\phi_p$, a $d_{x^2-y^2}$ gap.

For this model, the c-axis conductivity sum rule has the form[5]

$$\frac{2}{\pi e^2 d^2} \int_0^\infty \sigma_{1c}(\omega)d\omega = - <K_c> \tag{3}$$

where $d$ is the interlayer spacing, and $<K_c>$ is the c-axis kinetic energy per unit volume

$$<K_c> = \frac{<H_c>}{V} \tag{4}$$

If the change in $<K_c>$ between the normal and superconducting states is negligible, one has the usual relationship between the loss in the ($\omega > 0$) spectral weight of the conductivity in the superconducting state relative to the normal state and the c-axis penetration depth, $\lambda_c$[6,7]

$$\frac{c^2}{4\pi\lambda_c^2} = \frac{2}{\pi} \int_{0^+}^\infty d\omega(\sigma_{1c}^N(\omega) - \sigma_{1c}^S(\omega)) \tag{5}$$

Here $\sigma_{1c}^N$ and $\sigma_{1c}^S$ are the normal and superconducting c-axis conductivities, respectively. However, when the c-axis tunneling process is incoherent and the gap has a strong momentum dependence, the change in $<K_c>$ between the superconducting and normal states becomes important. Then Eqn.(5) is modified to

$$\frac{c^2}{4\pi\lambda_c^2} = \frac{2}{\pi} \int_{0^+}^\infty d\omega(\sigma_{1c}^N(\omega) - \sigma_{1c}^S(\omega))$$
$$- e^2 d^2(<K_c>^S - <K_c>^N) \tag{6}$$

For the case of a $d_{x^2-y^2}$ superconductor, if the tunneling process is diffuse, the Josephson coupling between the layers vanishes[2-4] and $\lambda_c$ is infinite. In this case, $\sigma_{1c}(\omega)$ is still suppressed when the gap is opened (see Fig. 2 of Ref. 4) but the change in the kinetic energy in Eqn.(6) cancels the change in the spectral weight, leading to an infinite $\lambda_c$. If the incoherent tunneling process is anisotropic, there will only be a partial cancellation, leading to a larger $\lambda_c$ than one would find using Eqn.(5). Here, we examine this effect for an impurity scattering model of the interlayer transport.

Taking $V_l$ to be weak, the first non-vanishing contribution to $<K_c>$, after averaging over impurities[4,8], is

$$\frac{4n_{imp}^c}{N_{ab}^2} \sum_{k,p} \overline{|V_{pk}|^2}T \sum_n \frac{(i\omega_n + \epsilon_p)(i\omega_n + \epsilon_k)}{[(i\omega_n)^2 - E_p^2][(i\omega_n)^2 - E_k^2]}$$
$$- \frac{4n_{imp}^c}{N_{ab}^2} \sum_{k,p} \overline{|V_{pk}|^2}T \sum_n \frac{\Delta_k\Delta_p}{[(i\omega_n)^2 - E_p^2][(i\omega_n)^2 - E_k^2]} \tag{7}$$

where $n_{imp}^c$ is the impurity concentration which causes c-axis transport, $N_{ab}$ is the number of sites in the $ab$ plane, $\omega_n = (2n+1)\pi T$, and we will take the impurity potential to have the separable form

$$\overline{|V_{pk}|^2} = |V_0|^2 + |V_1|^2 \cos 2\phi_k \cos 2\phi_p \tag{8}$$

Physically, the first term in Eqn.(7) is due to quasi-particle fluctuations between the layers, while the second term is due to superconducting pair fluctuations.

Setting $\Delta_k = 0$ in Eqn.(7) gives us $< K_c >^N$. Taking $\Delta_k = \Delta_0 \cos 2\phi_k$ gives $< K_c >^S$ for a $d_{x^2-y^2}$ superconductor. Thus we find that

$$< K_c >^S_{d_{x^2-y^2}} - < K_c >^N =$$

$$-16n^c_{imp}N^2(0)\,T\sum_n |V_0|^2\,[\frac{\omega_n^2}{\Delta_0^2+\omega_n^2}\mathbf{K}^2(\frac{\Delta_0}{\sqrt{\Delta_0^2+\omega_n^2}})-(\frac{\pi}{2})^2]$$

$$-16n^c_{imp}N^2(0)\,T\sum_n \{\frac{|V_1|^2}{\Delta_0^2(\Delta_0^2+\omega_n^2)}[\omega_n^2\mathbf{K}(\frac{\Delta_0}{\sqrt{\Delta_0^2+\omega_n^2}})$$

$$-(\Delta_0^2+\omega_n^2)\mathbf{E}(\frac{\Delta_0}{\sqrt{\Delta_0^2+\omega_n^2}})]^2\} \qquad (9)$$

where $N(0)$ is the bare single particle density of states, and $\mathbf{K}$ and $\mathbf{E}$ are complete elliptic integrals of the first and second kinds, respectively[9]. For $T \ll \Delta_0$ ,

$$< K_c >^S_{d_{x^2-y^2}} - < K_c >^N =$$

$$\frac{8n^c_{imp}N^2(0)}{\pi}\Delta_0\left(5.12|V_0|^2 - 2.37|V_1|^2\right)$$

$$+ \mathcal{O}[(\frac{T}{\Delta_0})^3\ln^2(\frac{T}{\Delta_0})] \qquad (10)$$

Then from Eqn.(6) we have

$$\frac{c^2}{4\pi\lambda_c^2} = \frac{2}{\pi}\int_{0+}^\infty d\omega(\sigma^N_{1c}(\omega) - \sigma^S_{1c}(\omega))$$

$$-\frac{8n^c_{imp}N^2(0)}{\pi}e^2d^2\Delta_0(5.12\,|V_0|^2\,-2.37\,|V_1|^2) \quad (11)$$

However, we know that when $|V_1|^2 = 0$ there is no pair transport and $\lambda_c$ becomes infinite. In this case, the $|V_0|^2$ term gives the difference between the area under $\sigma^N_{1c}(\omega,T)$ and $\sigma^S_{1c}(\omega,T)$ for $\omega > 0$ and there is no $\delta$-function contribution at $\omega = 0$. For $|V_1|^2$ small but non-vanishing, $\lambda_c$ becomes finite but *larger* than one would estimate from the missing spectral area $\sigma^N_{1c}(\omega,T) - \sigma^S_{1c}(\omega,T)$ for $\omega > 0$. If $|V_1|^2$ increases sufficiently so that $|V_1|^2 = 2.16\,|V_0|^2$ then there is no change between $< K_c >^S$ and $< K_c >^N$ and the correct $\lambda_c$ is obtained from the familiar sum rule, Eqn.(5).

Eqn.(18) of Ref.[4] gives a prediction for the c-axis penetration depth for the model we considered here. It is

$$\frac{c^2}{4\pi\lambda_c^2} \simeq 4\pi e^2 d^2 n^c_{imp}N^2(0)|V_1|^2\Delta_0(.48) \qquad (12)$$

This is the result one would obtain if a direct magnetic measurement of the penetration depth were made. Our eqn.(11) also gives a prediction for the c-axis penetration depth. However, eqn.(11) is the penetration depth inferred from a measurement of the conductivity. Our results show that for a momentum-dependent gap, the conductivity sum rule must be applied with care to determine the penetration depth. Our results show that,

for a momentum-dependent gap, there is a change in the c-axis kinetic energy between the normal and superconducting states; this change in kinetic energy must be taken into account in order to correctly obtain the penetration depth from the conductivity sum rule. A nieve application of the conductivity sum rule ( eqn.(5) ) would imply a penetration depth which is smaller or larger than what would be measured. From a correct application of the sum rule ( eqn.(6) ), the correct value of the penetration depth could be inferred. Eqn.(11) and eqn.(12) give the same value for the penetration depth. However, eqn.(11) is what one would use to infer the penetration depth from a measurement of the conductivity.

One can ask what has happened to the conductivity spectral weight. As Hirsch discussed[10], spectral weight can be transferred to or from higher bands which are not included in our simple interlayer hopping model. Note however, for $|V_1|^2 < 2.16\,|V_0|^2$ we have the opposite effect to that discussed by Hirsch for his model of hole superconductivity. That is, for the impurity model we have considered here, when the system goes into the superconducting state, if $|V_1|^2 < 2.16\,|V_0|^2$, spectral weight is transferred to higher bands and the true $\lambda_c$ is larger than one would obtain by simply determining the missing spectral weight according to Eqn.(5). Conversely, if $|V_1|^2 > 2.16\,|V_0|^2$ spectral weight is transferred down from higher bands and the true $\lambda_c$ is actually smaller than that given by Eqn.(5).

[1] C. C. Homes *et. al.*, Phys. Rev. Lett. **71**, 1645, (1993); Physica C **254**, 265, (1995). D. N. Basov *et. al.*, Phys. Rev. B **50**, 3511, (1994); Phys. Rev. B **52**, R13141, (1995).

[2] M. Graf *et. al.*, Phys. Rev. B **52**, 10588, (1995).

[3] R.J. Radtke, K. Levin, Physica C **250**, 282, (1995)

[4] P.J. Hirschfeld, S.M. Quinlan, D.J. Scalapino, Phys. Rev. B **55**, 12742, (1997).

[5] P.F. Maldague, Phys. Rev. B **16**, 2437, (1977).

[6] R.A. Ferrell and R.E. Glover, III, Phys. Rev. **109**, 1398, (1958).

[7] R.A. Ferrell and M. Tinkham, Phys. Rev. Lett. **2**, 331, (1959).

[8] G.D. Mahan, *Many-Particle Physics*, (Plenum, New York, 1981).

[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* , (Academic Press, San Diego, 1994).

[10] J. Hirsch, Journ. Phys. Chem. of Solids **54**, 1101, (1993).